

Curve Criterion

A function f of several variables satisfies $\lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x}) = L$
iff for all continuous space curves $\vec{r}(t)$
with $\lim_{t \rightarrow \infty} \vec{r}(t) = \vec{a}$ and $\vec{r}(t) \neq \vec{a}$ for all t

we have $\lim_{t \rightarrow \infty} f(\vec{r}(t)) = L$

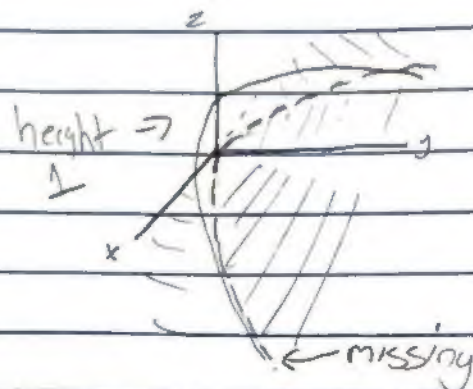
Idea: Find two curves $\vec{r}_0(t)$ and $\vec{r}_1(t)$ with $\lim_{t \rightarrow \infty} \vec{r}_i(t) = \vec{a}$
and $\lim_{t \rightarrow \infty} f(\vec{r}_0(t)) \neq \lim_{t \rightarrow \infty} f(\vec{r}_1(t))$.

Note: The collection of lines given by $\ell_{a,b}(t) = \vec{a} + t\langle a, b \rangle$
 a good collection of curves to start w/

$$\lim_{t \rightarrow 0} \ell_{a,b}(t) = \vec{a}$$

↖ This way isn't always good enough

Ex: Let $f(x,y) = \begin{cases} 1 & \text{if } y = x^2 \\ 0 & \text{otherwise} \end{cases}$



Claim: $\lim_{\vec{x} \rightarrow \vec{0}} f(\vec{x})$ does not exist

For lines $\ell_{a,b}(t)$: $\lim_{t \rightarrow 0^+} f(\ell_{a,b}(t)) = \lim_{t \rightarrow 0^+} f(at, bt)$

off of the origin (at, bt) satisfies $bt = (at)^2$ at

$\therefore f(at, bt) = 0$ for all but most once
(solve for t)
 finitely many times

$$\therefore \lim_{t \rightarrow 0^+} f(\ell_{a,b}(t)) = 0$$

On the other hand,

Letting $\vec{r}(t) = \langle t, t^2 \rangle$, we see $f(\vec{r}(t)) = f(t, t^2) = 1$
 for all t

$$\therefore \lim_{t \rightarrow 0} f(\vec{r}(t)) = \lim_{t \rightarrow 0} 1 = 1$$

Thus $\lim_{\vec{x} \rightarrow \vec{0}} f(\vec{x})$ does not exist \square

How can we show limits do exist?

A trick: Try polar coordinates

Ex: Does $\lim_{(x,y) \rightarrow 0,0} \frac{\sin(x^2+y^2)}{x^2+y^2}$ exist?

Sol: Convert limits to polar coordinates: $\begin{cases} x = r \cos(\theta) \\ y = r \sin(\theta) \end{cases}$

$$\lim_{(x,y) \rightarrow 0,0} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{r \rightarrow 0^+} \frac{\sin(r^2(\cos^2\theta + \sin^2\theta))}{r^2(\cos^2\theta + \sin^2\theta)} \quad \text{iff } r \rightarrow 0^+$$

→ if it exists

$$= \lim_{r \rightarrow 0^+} \frac{\sin(r^2(\cos^2\theta + \sin^2\theta))}{r^2(\cos^2\theta + \sin^2\theta)}$$

$$= \lim_{r \rightarrow 0^+} \frac{\sin(r^2)}{r^2} \Rightarrow \frac{0}{0} \text{ type}$$

$$\text{L'H} \quad \lim_{r \rightarrow 0^+} \frac{2r \cos(r^2)}{2r} = \lim_{r \rightarrow 0^+} \cos(r^2) = \cos(0^2) = 1$$

Ex Does $\lim_{(x,y) \rightarrow 0,0} \frac{x^2-y^2}{x^2+y^2}$ exist?

$$\text{Sol} \quad \lim_{(x,y) \rightarrow 0,0} \frac{x^2-y^2}{x^2+y^2} = \lim_{r \rightarrow 0^+} \frac{(r \cos \theta)^2 - (r \sin \theta)^2}{(r \cos \theta)^2 + (r \sin \theta)^2}$$

$$= \lim_{r \rightarrow 0^+} \frac{r^2(\cos^2\theta - \sin^2\theta)}{r^2(\cos^2\theta + \sin^2\theta)} = \lim_{r \rightarrow 0^+} \cos 2\theta = \cos(2\theta)$$

Depends

on θ

If we approach along angle $\theta = \pi/2$, we expect $\lim_{x \rightarrow 0} f(x) = \cos(2(\pi/2)) = -1$

If $\theta = 0$ $\cos(0) = 1$ $\therefore \lim_{x \rightarrow 0} \frac{x^2-y^2}{x^2+y^2} \text{ DNE}$

Def: A function f of n variables is continuous at $\vec{a} \in \text{dom}(f)$ when

$$\lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x}) = f(\vec{a})$$

f is continuous on set D when f is cts. at every member of D

Ex Every polynomial in n -variables is cts. on \mathbb{R}^n

Ex Every rational function of n -variables is cts. on its domain

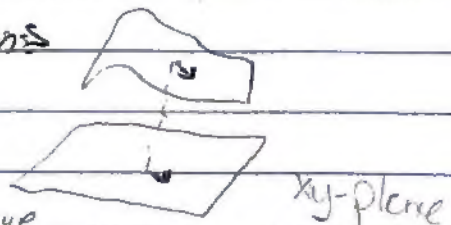
↳ $\frac{x^2 - y^2}{x^2 + y^2}$ is cts on its domain (everywhere but $(0,0)$)

Ex $\frac{\sin(x^2 + y^2)}{x^2 + y^2}$ is cts. everywhere but $(0,0)$ ↪ non-domain

OTOH $f(x,y) = \begin{cases} \frac{\sin(x^2 + y^2)}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$ is cts everywhere

Derivatives of Multivariable Functions

Idea: The derivative measures change in output from corresponding small change in input IN SOME DIRECTION



Def: Let f be a function of n variables and take \vec{u} a unit vector in \mathbb{R}^n . Let $\vec{a} \in \text{dom}(f)$

The directional derivative of f at \vec{a} in direction

of \vec{u} is $D_{\vec{u}} f(\vec{a}) = \lim_{h \rightarrow 0^+} \frac{f(\vec{a} + h\vec{u}) - f(\vec{a})}{h}$

Ex Compute the directional derivative of $f(x,y) = xy$ at $\vec{a} = \langle 1, 3 \rangle$ in direction $\vec{u} = \frac{1}{2} \langle \sqrt{2}, \sqrt{2} \rangle$

Sol: $D_{\vec{u}} f(\vec{a}) = \lim_{h \rightarrow 0^+} \frac{f(\vec{a} + h\vec{u}) - f(\vec{a})}{h}$

$$= \lim_{h \rightarrow 0^+} \frac{f(1 + \frac{\sqrt{2}}{2}h, 3 + \frac{\sqrt{2}}{2}h) - f(1, 3)}{h}$$

$$= \lim_{h \rightarrow 0^+} \frac{(1 + \frac{\sqrt{2}}{2}h)(3 + \frac{\sqrt{2}}{2}h) - 1 \cdot 3}{h}$$

$$= \lim_{h \rightarrow 0^+} \frac{3 + h(\frac{3\sqrt{2}}{2} + \frac{\sqrt{2}}{2}) + h^2 - 3}{h}$$

$$= \lim_{h \rightarrow 0^+} \frac{h(2\sqrt{2} + h)}{h} = \lim_{h \rightarrow 0^+} (2\sqrt{2} + h) = 2\sqrt{2} + 0 = 2\sqrt{2}$$

Exercise: Repeat exercise with $\vec{a} = \langle x, y \rangle$

Note: The directional derivative is very general

Def: Let F be a function of n -variable and let \vec{e}_k be the standard k -th basis vector in \mathbb{R}^n , i.e. $\vec{e}_k = \langle 0, 0, \dots, 1, \dots, 0 \rangle$
 \uparrow
 k -th position

The k -th partial derivative of F
 $D_{\vec{e}_k} f(\vec{a})$